The Ungar Game on Various Lattices

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Lattice Preliminaries: Covering Relations

Posets P

- A set P with a binary relation $\leq$.
- Not every pair of elements in P needs to be comparable.

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Example: Poset

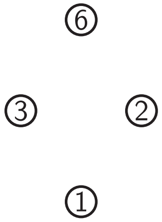
- Set: $\{1, 2, 3, 6\}$
- Order relation: $u \leq v$ if u divides v.
- $1 \leq 2$ and $1 \leq 6$.
- $1 \lessdot 2$, but 6 does not cover 1.

Lattice Preliminaries: Poset

Poset Diagram

- Nodes represent elements.
- If $u < v$, then v is located higher than u.

Example: Poset Diagram



- A poset with elements $\{1, 2, 3, 6\}$ and $u \leq v$ if u divides v.

Lattice Preliminaries: Poset

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- Two nodes u and v are connected by an edge if v covers u

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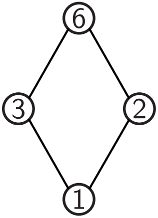

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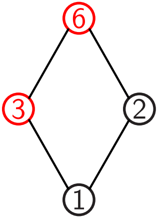
Example: Poset Diagram



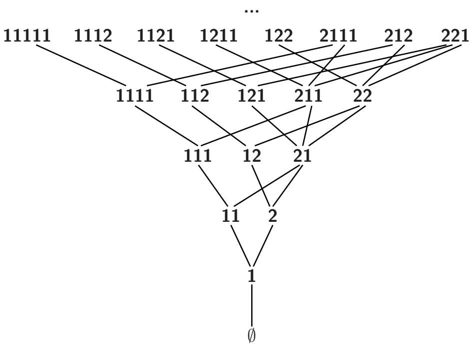
- A poset with elements $\{1, 2, 3, 6\}$ and $u \leq v$ if u divides v. Here 6 covers 3.

Lattice Preliminaries: Covering Relations

Young-Fibonnacci Lattice $\mathbb{YF}$

- The elements in $\mathbb{YF}$ are the words of the alphabets $\{1, 2\}$.

Example: Young-Fibonnacci Lattice



Lattice Preliminaries: Covering Relations

Young-Fibonnacci Lattice $\mathbb{YF}$

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- $u \lessdot v$ iff

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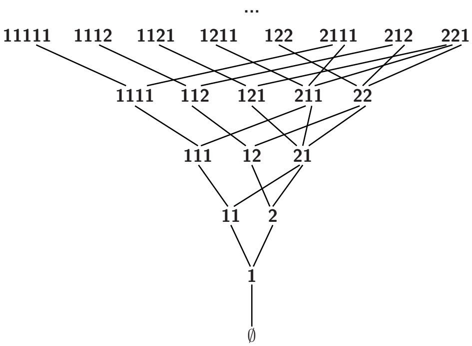

Lattice Preliminaries: Covering Relations

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  - v is u with the leftmost 1 replaced with a 2, or

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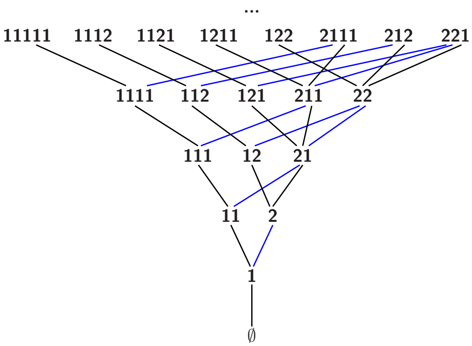

Lattice Preliminaries: Covering Relations

Young-Fibonnacci Lattice $\mathbb{YF}$

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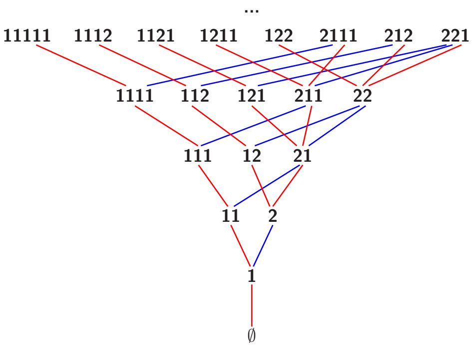
- $u \lessdot v$ iff
  - v is u with the leftmost 1 replaced with a 2, or
  - v is u with a 1 inserted anywhere left of the leftmost 1 in u.

Example: Young-Fibonnacci Lattice

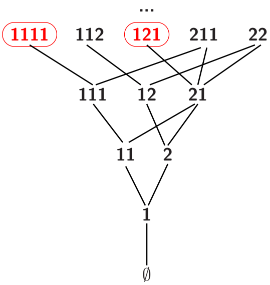


Lattice Preliminaries: Meet-Semilattices

Meet-Semilattices P

- Posets in which any two elements $u, v \in P$ have a greatest lower bound $u \wedge v \in P$ called their **meet**.

Example: Young-Fibonnacci Lattice



- Let u and v be 1111 and 121, respectively.

Lattice Preliminaries: Meet-Semilattices

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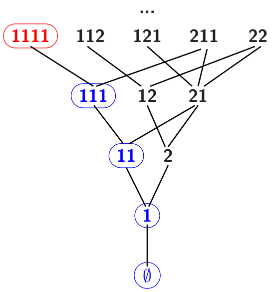

- The elements that are ≤ 1111 are colored blue.

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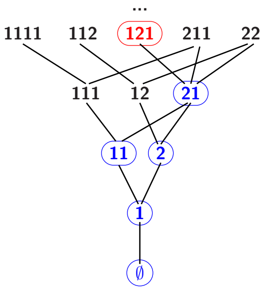
Example: Young-Fibonnacci Lattice



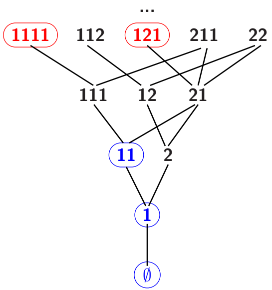
- Now, the elements that are ≤ 121 are colored blue.

Lattice Preliminaries: Meet-Semilattices

Meet-Semilattices P

- Posets in which any two elements $u, v \in P$ have a greatest lower bound $u \wedge v \in P$ called their **meet**.

Example: Young-Fibonnacci Lattice



- Now, the elements that are both ≤ 1111 and ≤ 121 are colored blue.

Lattice Preliminaries: Meet-Semilattices

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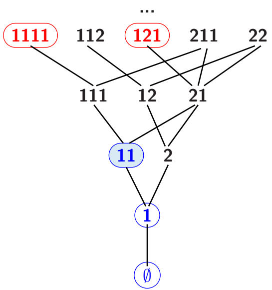


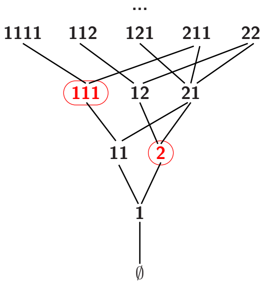
- The greatest of such elements—the meet—is shaded, i.e. 11.

Lattice Preliminaries: Lattices

Lattices P

- Meet-semilattices in which any two elements $u, v \in P$ have a least upper bound $u \vee v \in P$ called their **join**.

Example: Young-Fibonnaci Lattice



- Let $u = 111$ and $v = 2$.

Lattice Preliminaries: Lattices

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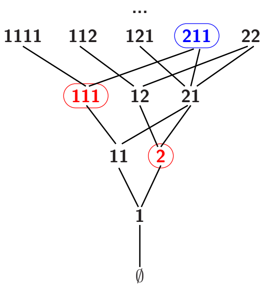
Example: Young-Fibonnacci Lattice



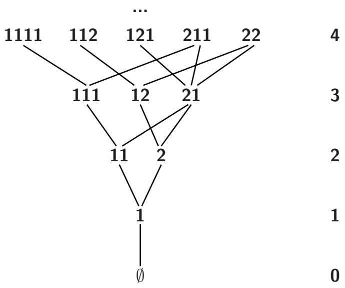
- The join of 111 and 2 is 211.

Lattice Preliminaries: Graded Lattices

Graded Lattices

- A *graded lattice* P has a rank function ρ such that $\rho(\hat{0}) = 0$ and $\rho(u) + 1 = \rho(v)$ if $u \lessdot v$.

Example: Young-Fibonnacci Lattice Up to Rank 4

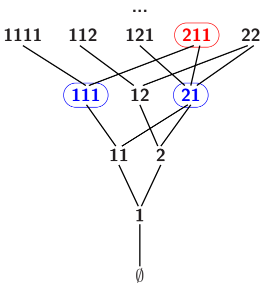


Ungar Moves

Ungar Move (Defant and Li, 2023)

- An operation that sends $v \in P$ to the meet of the set $\{v\} \cup T$, where T is some subset of the elements that v covers.

Example: Ungar Moves on the Young-Fibonnacci Lattice



- Let $v = 211$, where v covers 111 and 21.

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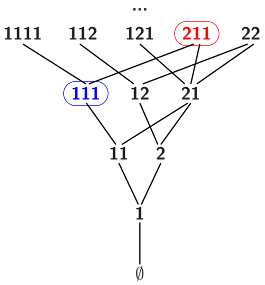

- If $T = \{111\}$, the meet of $\{v\} \cup T$ is 111.

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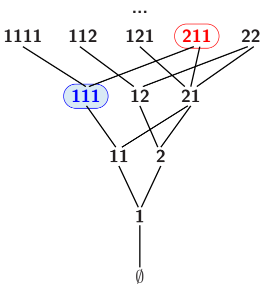

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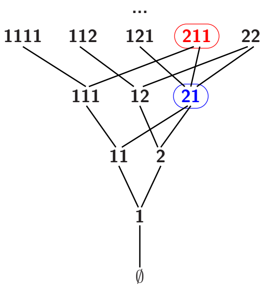

- If $T = \{21\}$, the meet of $\{v\} \cup T$ is 21.

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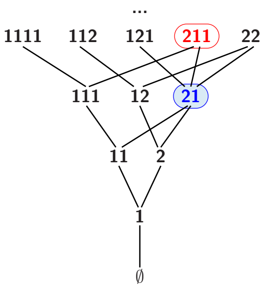

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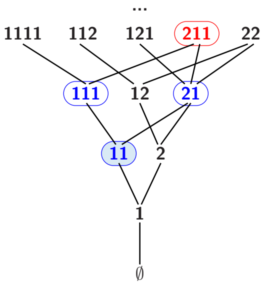

- If $T = \{111, 21\}$, then the meet of $\{v\} \cup T$ is 11.

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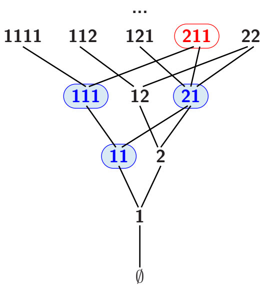
Example: Ungar Moves on the Young-Fibonnacci Lattice



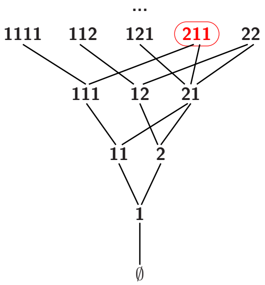
- Therefore, the set of Ungar moves of 211 is $\{111, 21, 11\}$.

The Ungar Game

The Ungar Game (Defant, Kravitz& Williams, 2024)

- Alternates between Atniss and Eeta in making Ungar moves starting from an element of P until the player with no possible moves loses.

Example: The Ungar Game on the Young-Fibonnacci Lattice



- Let us start from 211.

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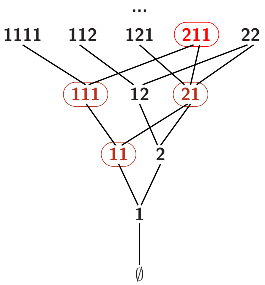

- The set of Ungar moves Atniss can make is $\{111, 21, 11\}$.

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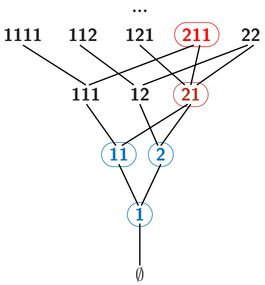

- Suppose Atniss goes to 21. Eeta's Ungar moves are $\{11, 2, 1\}$.

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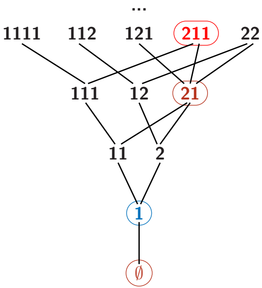

- Suppose Eeta goes to 1. Atniss must go to $\emptyset$. Thus, Atniss wins.

The Ungar Game Strategy

Atniss and Eeta Wins

- An element $v \in P$ is either an Atniss (belongs to $\mathbf{A}(L)$) or an Eeta win (belongs to $\mathbf{E}(L)$).

The Ungar Game Strategy

Atniss and Eeta Wins

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Strategy

- An element $v \in P$ is an Eeta win iff every element of $\mathrm{Ung}(v) \setminus \{v\}$ is an Atniss win. Otherwise, $v \in \mathbf{A}(L)$.

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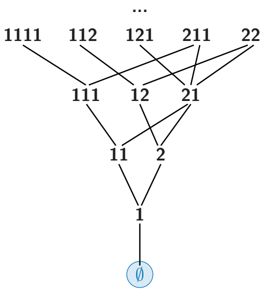

- The lowest element $\emptyset$ is an Eeta win, colored in blue.

The Ungar Game Strategy

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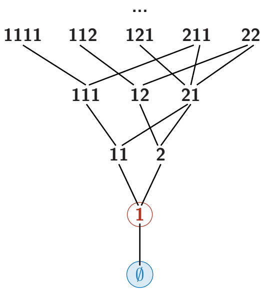

- The element 1 is an Atniss win, colored in red.

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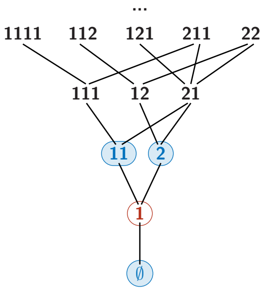
Example: The Ungar Game on the Young-Fibonnacci Lattice



- The elements 11 and 2 are Eeta wins.

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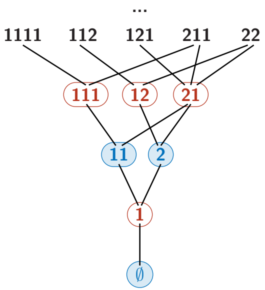

- The elements 111, 12, and 21 are Atniss wins.

The Ungar Game Strategy

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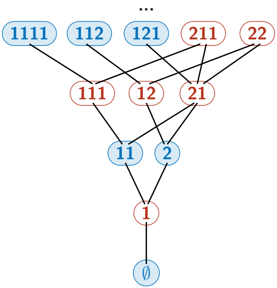

- The $\mathbb{YF}$ lattice with Atniss wins in red and Eeta wins in blue.

Conjectures on the Ungar Games

Conjectures (Defant, Kravitz & Williams, 2024)

- Young-Fibonacci Lattice: enumeration of Eeta wins
- Lattice of order ideals in the shifted staircases: complete characterization of Eeta wins

Theorems on the Ungar Games

Theorems (C. & G., 2024)

- Young-Fibonacci lattice: complete characterization and enumeration of Eeta wins
- Lattice of order ideals in the shifted staircases: complete characterization and enumeration of Eeta wins
- $J(\mathbb{N}^3)$ lattice: complete characterization of Eeta wins
- $J(\mathbb{N}^k)$ lattice: characterization of Eeta wins
- $\mathrm{Weak}(B_n)$ lattice: characterization of Eeta wins

Theorems on the Ungar Games

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- $J(\mathbb{N}^3)$ lattice: complete characterization of Eeta wins
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Eeta Wins on the Young-Fibonnacci Lattice

Theorem (C. & G., 2024).

- For rank $r \geq 2$, an element $v \in \mathbb{YF}$ of rank r is an Eeta win iff
  - $v_{1:|v|-1} = 11 \cdots 1$ and the number of 1s in v is even, or
  - $v_{1:|v|-1} \neq 11 \cdots 1$ and the number of 1s to the left of the leftmost 2 in v is odd.

Eeta Wins on the Young-Fibonnacci Lattice

Theorem (C. & G., 2024).

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Corollary (C. & G., 2024).

- For $r \geq 2$, it holds that $|\mathbf{E}_r(\mathbb{YF})| = f_{r-2} + (-1)^r$.

Shifted Staircases

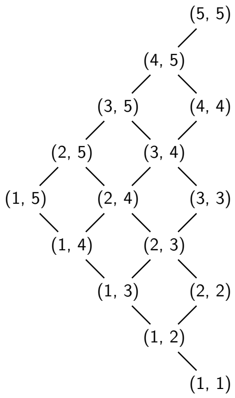
The n^{th} Shifted Staircase SS_n

- All pairs $(x, y) \in \mathbb{N}^2$ such that $1 \leq x \leq y \leq n$.
- $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2$ and $x_2 \leq y_2$.

Order Ideal

- A subposet $I \subseteq P$ such that for any element $u \in P$ and any element $v \in I$, if $u \leq v$, then $u \in I$.

Example: 5^{th} Shifted Staircase



The 5^{th} shifted staircase.

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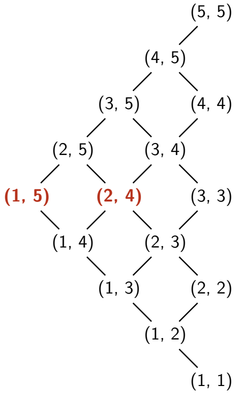

Let $(1, 5), (2, 4) \in I$.

Shifted Staircases

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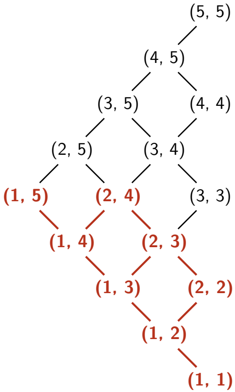


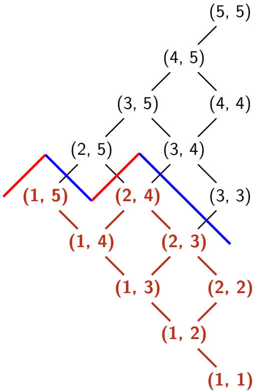
The red elements form an order ideal I.

Shifted Staircases

A Natural Bijection

- There exists a bijection between the order ideals and binary strings defined by up and down steps on path.

Example: 5^{th} Shifted Staircase



The path of I with binary string 10100.

Order Ideals in Shifted Staircases

Lattice of Order Ideals

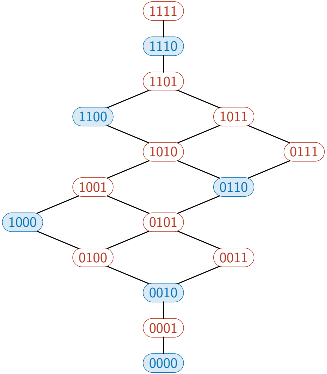
- Let $J(\mathrm{SS}_n)$ be the lattice of the order ideals of the n^{th} shifted staircase SS_n ordered by containment.

Covering Relations

$u \lessdot v$ if

- A 10 in v is a 01 in u, or
- 1 is the last digit of v and is a 0 in u.

Example: $J(\mathrm{SS}_4)$ Lattice



The lattice $J(\mathrm{SS}_4)$ with Atniss wins unshaded and colored red and Eeta wins shaded and colored blue.

Eeta Wins on the Order Ideals in Shifted Staircases

Theorem (C. & G., 2024).

An order ideal $v \in J(\mathrm{SS}_n)$ with binary representation $s \in \{0, 1\}^n$ is an Eeta win iff

- $s_{|s|} = 0$, and
- there are no odd-length sequences of 1s followed by an odd-length sequence of 0s in $s_{1:|s|-1}$.

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Corollary (C. & G., 2024).

- For all n, it holds that

$$|\mathbf{E}(J(\mathrm{SS}_n))| = \sum_{k \geq 0}^{\lfloor \frac{n}{2} \rfloor} 2^k \binom{k+2}{n-2k}$$

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